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Interval Observers for Estimating Unknown Inputs in Discrete Time-Invariant Dynamic Systems

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Abstract—This paper considers discrete time-invariant systems (with constant parameters) described by dynamic models in the presence of exogenous disturbances. For such systems, the problem of estimating unknown inputs using interval observers is studied. The solution is based on a minimal-dimension disturbance-insensitive model of the original system. For this model, an interval observer is designed and then used to estimate the unknown inputs. The theoretical results are illustrated with a practical example.

Keywords: linear systems, unknown inputs, interval observers, estimation

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1. INTRODUCTION AND PROBLEM STATEMENT

The problem of estimating unknown inputs of dynamic systems has theoretical and applied significance, particularly for determining the changes in system parameters due to arising faults to parry these changes or the level of exogenous disturbances by a control system. For systems described by continuous-time models, in the past twenty-five years, this problem has been solved using sliding mode observers [1–3].

On the other hand, interval observers have been applied in recent years to estimate the components of the state vector of dynamic systems. At each time instant, such observers produce an estimate of the set of admissible values of the state vector (or a given linear function of this vector) for various classes of dynamic systems with uncertainty. Detailed reviews of the contemporary results were provided in [4–6]; solutions for various classes of systems (continuous, discrete, hybrid, and delayed), as well as practical applications, can be found in [7–14]. A peculiarity is that interval observers involve simple means to consider various types of uncertainty in systems (exogenous disturbances, measurement noises, and parametric uncertainties) and generate an interval surely containing the values of the state vector (or a given linear function of this vector).

Interval observers were proposed in [15] to estimate unknown inputs in discrete-time linear dynamic systems. This method is an alternative to sliding mode observers, which become ineffective in the discrete case.

In this paper, the approach introduced in [15] is further developed to estimate unknown inputs in discrete-time linear and nonlinear dynamic systems subjected to exogenous disturbances. In contrast to [15], the system is not required to be minimum-phase; moreover, the approach proposed therein involves several complex transformations of the system, in particular, reduction to a descriptor form, which is not required below. By analogy with [5–13], the interval observer in [15] was designed for the original transformed system and therefore has full dimension. In contrast, the solution presented below uses a reduced model of the original system, simplifying the observer design procedure and making the resulting interval observer insensitive to exogenous disturbances. In some cases, this property ensures an exact estimate even in the presence of exogenous disturbances. In addition, we overcome the matching condition requirement imposed in [15], which restricts estimation capabilities.

2. MAIN MODELS AND RELATIONS

Consider a system described by the discrete-time nonlinear model

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + C\Psi(x(t), u(t)) + Dd(t) + L\rho(t), \\ y(t) &= Hx(t), \end{aligned} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$ are the state vector, control input, and output, respectively; F, G, H, C, and L are known constant matrices; $\rho(t) \in \mathbb{R}^p$ is an unknown bounded time-varying function that describes disturbances affecting the system; the term $Dd(t)$ is an unknown input to be estimated. In particular, such an input may manifest a system fault. By assumption, $d(t)$ is a scalar bounded function, i.e., $\underline{d} \leqslant d(t) \leqslant \overline{d}$ for all $t = 0, 1, 2, \ldots$ with given values $\underline{d}$ and $\overline{d}$. The nonlinear component $\Psi(x, u)$ is represented as

$$\Psi(x, u) = \begin{pmatrix} \varphi_1(A_1 x, u) \\ \cdots \\ \varphi_q(A_q x, u) \end{pmatrix},$$

where $A_1, \ldots, A_q$ are known constant row matrices, and $\varphi_1, \ldots, \varphi_q$ are arbitrary nonlinear functions.

Remark 1. By analogy with [15], the matching condition $\operatorname{rank}(HD) = \operatorname{rank}(D)$ is supposed to hold in the main part of this work, except for Section 7. This means that the unknown input $d(t)$ is included in the equations for the state vector components measured by system sensors. In contrast to [15], where $d(t)$ was an arbitrary-dimension function, we first assume that $d(t)$ is a scalar and then eliminate this constraint in Section 8.

The problem under consideration is first solved in the linear case, when $C = 0$. As stated, a solution will be found based on a reduced model of the original system. For this purpose, let us briefly recall the results of [14, 16]. In the cited papers, a linear minimal-dimension disturbance-insensitive model of system (2.1) was built to estimate some variable $y_*(t) = R_* y(t)$ for the matrix R_* to be determined during the solution procedure. Such a model is described by the equations

$$\begin{aligned} x_*(t+1) &= F_* x_*(t) + J_* y(t) + G_* u(t) + D_* d(t), \\ y_*(t) &= H_* x_*(t) = R_* y(t), \end{aligned} \tag{2.2}$$

where $x_*(t) \in \mathbb{R}^k$, $k < n$ denotes the model dimension and the matrices F_*, G_*, J_*, and H_* have to be determined. By assumption, there exists a matrix Φ such that $x_*(t) = \Phi x(t)$. According to [14, 16], the model matrices satisfy the equations

$$\Phi F = F_* \Phi + J_* H, \quad H_* \Phi = R_* H, \quad \Phi G = G_*, \quad \Phi D = D_*, \quad \Phi L = 0. \tag{2.3}$$

A necessary condition for building such a system is the inequality

$$\operatorname{rank} \begin{pmatrix} H \\ L_0 \end{pmatrix} < \operatorname{rank}(H) + \operatorname{rank}(L_0), \tag{2.4}$$

where L_0 is a matrix of maximal rank such that $L_0 L = 0$. Indeed, the condition $\Phi L = 0$ can be written as $\Phi = NL_0$ for some matrix N. Since $H_*\Phi = R_*H$, we have $H_*NL_0 = R_*H$, holding under condition (2.4). If the latter fails, then the disturbance-insensitive system estimating the variable $y_*(t)$ cannot be built. Below we will suppose the validity of condition (2.4).

3. BUILDING THE DISTURBANCE-INSENSITIVE MODEL

In the general case, the matrices F_* and H_* are given in the identification canonical form:

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad H_* = (1 \ \ 0 \ \ \dots \ \ 0). \tag{3.1}$$

If system (2.2) is observable, it can always be reduced to this form [17]. Otherwise, it can be reduced to a form with an observable subsystem; then the matrices of this subsystem can be found in the form (3.1) [17]. The problem is solved based on the equation

$$(\ R_* \ \ -J_{*1} \ \ \dots \ \ -J_{*k}\ )(W^{(k)} \ \ L^{(k)}) = 0, \tag{3.2}$$

where J_{*i} denotes the ith row of the matrix J_*,

$$W^{(k)} = \begin{pmatrix} HF^k \\ HF^{k-1} \\ \dots \\ H \end{pmatrix}, \quad L^{(k)} = \begin{pmatrix} HL & HFL & \dots & HF^{k-1}L \\ 0 & HL & \dots & HF^{k-2}L \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad k = 1, 2, \dots.$$

The matrix $W^{(k)}$ serves to build model (2.2) whereas the matrix $L^{(k)}$ ensures its insensitivity to the disturbances, i.e., the condition $\Phi L = 0$. Equation (3.2) has a nontrivial solution if

$$\operatorname{rank}(W^{(k)} \ \ L^{(k)}) < l(k+1) \tag{3.3}$$

because, in this case, the rows of the compound matrix $(W^{(k)} \ \ L^{(k)})$ are linearly dependent, which ensures the existence of a solution.

To build the model, it is necessary to determine a minimum number k from (3.3), starting from $k = 1$, and the row $(\ R_* \ \ -J_{*1} \ \ \dots \ \ -J_{*k}\ )$ from (3.2); next, using the relations

$$R_*H = \Phi_1, \quad \Phi_i F = \Phi_{i+1} + J_{*i}H, \quad i = 1, \dots, k-1, \quad \Phi_k F = J_{*k}H,$$

obtained from (2.3) and (3.1), the matrix Φ is constructed; finally, G_* is found from (2.3), where Φ_i denotes the ith row of the matrix Φ. Thus, the linear model (2.2) is built; by assumption, $D_* \neq 0$.

If equation (3.2) has no solution for all $k < n$, then the disturbance-insensitive model does not exist. In this case, robust methods can be used; for example, see [18]. However, the resulting estimate will be inexact.

4. PROBLEM SOLUTION

For the sake of simplicity, we begin with the assumption $k = 1$. In other words, it is possible to build a one-dimensional model described by the equations

$$\begin{aligned} x_*(t+1) &= J_*y(t) + G_*u(t) + D_*d(t), \\ y_*(t) &= x_*(t). \end{aligned} \tag{4.1}$$

(Under this assumption, $F_* = 0$ and $H_* = 1$.)

Proposition 1. *Model* (4.1) *exists under two conditions:*

$$\operatorname{rank}\begin{pmatrix} HF \\ H \end{pmatrix} < \operatorname{rank}(HF) + \operatorname{rank}(H),$$

$$R_* HL = 0.$$

Proof. For $k = 1$, from (3.2) it follows that

$$\begin{pmatrix} R_* & -J_* \end{pmatrix} \begin{pmatrix} HF & HL \\ H & 0 \end{pmatrix} = 0,$$

which implies the equality $\begin{pmatrix} R_* & -J_* \end{pmatrix} \begin{pmatrix} HF \\ H \end{pmatrix} = 0$. It holds under the first condition. The second condition is immediate from the previous equality.

If at least one of these conditions fails, the model will be multidimensional. An additional condition is the fault sensitivity requirement $\Phi D = D_* \neq 0$, represented as $R_* HD \neq 0$.

Remark 2. Since the values of all variables in model (4.1), except for $d(t)$, are measurable, the latter can be determined from the simple relation $D_* d(t) = y_*(t+1) - J_* y(t) - G_* u(t)$. Considering the canonical form of the matrix F_* and the resulting model, similar relations for the variable $d(t)$ can be derived by introducing time shifts for the variable $y_*(t)$ and substituting some equations into others (as is done in Section 7). For $k > 2$, these relations will be quite cumbersome, and using interval observers allows significantly simplifying them. Therefore, despite the obvious solution for $k = 1$, we will illustrate the application of such observers in order to extend the result to the general case.

An interval observer is given by

$$\begin{aligned} \underline{x}_*(t+1) &= G_* u(t) + J_* y(t) + D_*^+ \underline{d} - D_*^- \overline{d} + g\underline{e}(t), \\ \overline{x}_*(t+1) &= G_* u(t) + J_* y(t) + D_*^+ \overline{d} - D_*^- \underline{d} + g\overline{e}(t), \\ \underline{y}_*(t) &= \underline{x}_*(t), \quad \overline{y}_*(t) = \overline{x}_*(t), \end{aligned} \tag{4.2}$$

where $A^+ = \max(0, A)$ and $A^- = A^+ - A$ for an arbitrary matrix A (clearly, $A^+ \geqslant 0$ and $A^- \geqslant 0$); the coefficient g is assigned to ensure the observer's stability; finally, $\underline{e}(t) = y_*(t) - \underline{y}_*(t)$ and $\overline{e}(t) = y_*(t) - \overline{y}_*(t)$.

For the interval observer, the condition $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$ implies $\underline{x}_*(t) \leqslant x_*(t) \leqslant \overline{x}_*(t)$ for all $t \geqslant 0$ [4]; hence, there exists a number $\alpha(t) \geqslant 0$ such that, for all $t \geqslant 0$,

$$x_*(t) = \underline{x}_*(t) + \alpha(t)(\overline{x}_*(t) - \underline{x}_*(t)), \tag{4.3}$$

or

$$x_*(t) = \alpha(t)\overline{x}_*(t) + (1 - \alpha(t))\underline{x}_*(t). \tag{4.4}$$

Let us determine the value $\alpha(t+1)$. For $t := t+1$, from (4.3) it follows that

$$R_* y(t+1) = x_*(t+1) = \underline{x}_*(t+1) + \alpha(t+1)(\overline{x}_*(t+1) - \underline{x}_*(t+1)),$$

yielding

$$\alpha(t+1) = \frac{R_* y(t+1) - \underline{x}_*(t+1)}{\overline{x}_*(t+1) - \underline{x}_*(t+1)}. \tag{4.5}$$

Note that all values in this formula are measurable.

Theorem 1. *The function $d(t)$ is estimated by*

$$\begin{aligned}\hat{d}(t) = D_*^{-1}(g(\alpha(t+1)\underline{e}(t) + (1-\alpha(t+1))\overline{e}(t)) \\ + \alpha(t+1)(D_*^+\underline{d} - D_*^-\overline{d}) + (1-\alpha(t+1))(D_*^+\overline{d} - D_*^-\underline{d})),\end{aligned} \tag{4.6}$$

where the coefficient $\alpha(t+1)$ is given by (4.5).

Proof. In the expression (4.4) written for $t := t+1$, we replace the variables $x_*(t+1)$, $\underline{x}_*(t+1)$ and $\overline{x}_*(t+1)$ with the right-hand sides of equations (4.1) and (4.2), respectively:

$$\begin{aligned}J_*y(t) + G_*u(t) + D_*d(t) = \alpha(t+1)(G_*u(t) + J_*y(t) + D_*^+\underline{d} - D_*^-\overline{d} + g\underline{e}(t)) \\ + (1-\alpha(t+1))(G_*u(t) + J_*y(t) + D_*^+\overline{d} - D_*^-\underline{d} + g\overline{e}(t)).\end{aligned} \tag{4.7}$$

Standard transformations on the right-hand side lead to

$$\begin{aligned}J_*y(t) + G_*u(t) + D_*d(t) = g(\alpha(t+1)\underline{e}(t) + (1-\alpha(t+1))\overline{e}(t)) + G_*u(t) + J_*y(t) \\ + \alpha(t+1)(D_*^+\underline{d} - D_*^-\overline{d}) + (1-\alpha(t+1))(D_*^+\overline{d} - D_*^-\underline{d}),\end{aligned}$$

which finally gives (4.6).

Remark 3. Due to the canonical form of the matrix F_*, the observer (4.2) is stable even without introducing the feedbacks $g\underline{e}(t)$ and $g\overline{e}(t)$; the latter are necessary only to obtain the estimate $\hat{d}(t)$, and (in this case) it will be exact for all $t > 0$.

5. THE MULTIDIMENSIONAL MODEL

In the case $k > 1$, model (2.2) takes the form

$$\begin{aligned}x_{*1}(t+1) &= x_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + D_{*1}d(t), \\ x_{**}(t+1) &= F_{**}x_{**}(t) + J_{**}y(t) + G_{**}u(t) + D_{**}d(t), \\ y_*(t) &= x_{*1}(t),\end{aligned}$$

where x_{*i} denotes the ith component of the vector x_*, $x_{**} = (x_{*2}, \ldots, x_{*k})$; J_{**}, G_{**}, and D_{**} are the matrices J_*, G_*, and D_*, respectively, with the removed first rows; F_{**} is the matrix (3.1) of dimensions $(k-1)\times(k-1)$. By assumption, $D_{*1} \neq 0$ and $D_{**} = 0$, i.e., the fault figures only in the first equation. The interval observer has the form

$$\begin{aligned}\underline{x}_{*1}(t+1) &= \hat{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + D_{*1}^+\underline{d} - D_{*1}^-\overline{d} + g_1\underline{e}(t), \\ \hat{x}_{**}(t+1) &= F_{**}\hat{x}_{**}(t) + J_{**}y(t) + G_{**}u(t) + g_{**}\underline{e}(t), \\ \overline{x}_{*1}(t+1) &= \tilde{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + D_{*1}^+\overline{d} - D_{*1}^-\underline{d} + g_1\overline{e}(t), \\ \tilde{x}_{**}(t+1) &= F_{**}\tilde{x}_{**}(t) + J_{**}y(t) + G_{**}u(t) + g_{**}\overline{e}(t), \\ \underline{y}_*(t) &= \underline{x}_{*1}(t), \quad \overline{y}_*(t) = \overline{x}_{*1}(t),\end{aligned} \tag{5.1}$$

i.e., only the first component of the vector is described by an interval. Here $\hat{x}_{**}(t)$ and $\tilde{x}_{**}(t)$ are vectors of dimension $(k-1)$, the matrix $(g_1\ g_{**}^T)^T$ is assigned for ensuring the observer's stability. (This can always be done for the canonical form (3.1).)

In view of (5.1), the expression (4.7) is modified as follows:

$$\begin{aligned}x_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + D_{*1}d(t) \\ = \alpha(t+1)(\tilde{x}_{*2}(t) + G_{*1}u(t) + J_{*1}y(t) + D_{*1}^+\underline{d} - D_{*1}^-\overline{d} + g_1\underline{e}(t)) \\ + (1-\alpha(t+1))(\hat{x}_{*2}(t) + G_{*1}u(t) + J_{*1}y(t) + D_{*1}^+\overline{d} - D_{*1}^-\underline{d} + g_1\overline{e}(t)).\end{aligned}$$

Here $\alpha(t+1)$ is determined for the first components of the state vectors by analogy with (4.5):

$$\alpha(t+1) = \frac{R_* y(t+1) - \underline{x}_{*1}(t+1)}{\overline{x}_{*1}(t+1) - \underline{x}_{*1}(t+1)}.$$

Standard transformations yield

$$\begin{aligned} D_{*1}d(t) &= g_1(\alpha(t+1)\underline{e}(t) + (1-\alpha(t+1))\overline{e}(t)) \\ &+ \alpha(t+1)(D_{*1}^+\underline{d} - D_{*1}^-\overline{d}) + (1-\alpha(t+1))(D_{*1}^+\overline{d} - D_{*1}^-\underline{d}) \\ &+ \alpha(t+1)\tilde{x}_{*2}(t) + (1-\alpha(t+1))\hat{x}_{*2}(t) - x_{*2}(t). \end{aligned} \tag{5.2}$$

Since the observer is stable, we have $\tilde{x}_{*2}(t) \to x_{*2}(t)$ and $\hat{x}_{*2}(t) \to x_{*2}(t)$ and, consequently, $\alpha(t+1)\tilde{x}_{*2}(t) + (1-\alpha(t+1))\hat{x}_{*2}(t) - x_{*2}(t) \to 0$. As a result, the estimate $\hat{d}(t)$ given by (4.6) with $D_* = D_{*1}$ satisfies $\hat{d}(t) \to d(t)$.

Remark 4. According to these relations, the use of an interval observer eliminates the effect of all components of the vector $x_{**} = (x_{*2}, \ldots, x_{*k})$ on the final result.

6. CONSIDERATION OF THE NONLINEARITIES

The model design method in the nonlinear case is based on the linear model built previously. The idea is to use the matrix Φ found for the linear model and analyze the possibility of transforming the argument of the nonlinear component $\Psi(x,u)$. This is done as follows; for details, see [18].

We calculate the matrix $C_* = \Phi C$ and determine the numbers $j_i, \ldots, j_s$ of its nonzero columns. Next, it is necessary to verify the condition

$$\operatorname{rank}\begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \Phi \\ H \\ A' \end{pmatrix}, \tag{6.1}$$

where the matrix A' consists of the matrices $A_1, \ldots, A_q$ with numbers $j_i, \ldots, j_s$ as rows. If this condition holds, the argument transformation is possible. After that, we build the nonlinear component

$$\Psi_*(x_*, y, u) = \begin{pmatrix} \varphi_{j_1}(A_{*j_1}z, u) \\ \cdots \\ \varphi_{j_s}(A_{*j_s}z, u) \end{pmatrix},$$

where $z = \begin{pmatrix} x_* \\ y \end{pmatrix}$ and the row matrices $A_{*j_1}, \ldots, A_{*j_s}$ are found from the linear equations

$$A_j = A_{*,j}\begin{pmatrix} \Phi \\ H \end{pmatrix}, \quad j = j_i, \ldots, j_s.$$

This component is added to the linear model (2.2). If condition (6.1) fails, it is necessary to find another solution of equation (3.2) given the same or increased dimension k and repeat the described procedure with the new matrix Φ. In the general case, the nonlinear model is described by the equation

$$\begin{aligned} &\dot{x}_*(t) = F_* x_*(t) + J_* y(t) + G_* u(t) + C_* \Psi_*(x_*(t), y(t), u(t)) + D_* d(t), \\ &y_*(t) = H_* x_*(t). \end{aligned}$$

If the model is one-dimensional, its nonlinear component will depend on $y_*(t) = x_*(t)$, and the interval observer will appear in the model in the same way. As is easily observed, in this case, the exact estimate will be given by (4.6).

In view of the assumption $D_{*1} \neq 0$ and $D_{**} = 0$, we obtain the following descriptions in the multidimensional case:

—for the model,

$$\begin{aligned} x_{*1}(t+1) &= x_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + C_{*1}\Psi_{*1}(y_*(t), x_{**}(t), y(t), u(t)) \\ &\quad + D_{*1}d(t), \\ x_{**}(t+1) &= F_{**}x_{**}(t) + J_{**}y(t) + G_{**}u(t) + C_{**}\Psi_{**}(y_*(t), x_{**}(t), y(t), u(t)); \end{aligned}$$

—for the interval observer,

$$\begin{aligned} \underline{x}_{*1}(t+1) &= \hat{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + C_{*1}\Psi_{*1}(y_*(t), \hat{x}_{**}(t), y(t), u(t)) \\ &\quad + D^+_{*1}\underline{d} - D^-_{*1}\overline{d} + g_1\underline{e}(t), \\ \hat{x}_{**}(t+1) &= F_{**}\hat{x}_{**}(t) + J_{**}y(t) + G_{**}u(t) + C_{**}\Psi_{**}(y_*(t), \hat{x}_{**}(t), y(t), u(t)) \\ &\quad + g_{**}\underline{e}(t), \\ \overline{x}_{*1}(t+1) &= \tilde{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + C_{*1}\Psi_{*1}(y_*(t), \tilde{x}_{**}(t), y(t), u(t)) \\ &\quad + D^+_{*1}\overline{d} - D^-_{*1}\underline{d} + g_1\overline{e}(t), \\ \tilde{x}_{**}(t+1) &= F_{**}\tilde{x}_{**}(t) + J_{**}y(t) + G_{**}u(t) + C_{**}\Psi_{**}(y_*(t), \tilde{x}_{**}(t), y(t), u(t)) \\ &\quad + g_{**}\overline{e}(t), \\ \underline{y}_*(t) &= \underline{x}_{*1}(t), \quad \overline{y}_*(t) = \overline{x}_{*1}(t), \end{aligned}$$

where the matrices C_{*1} and C_{**} and the functions Ψ_{*1} and Ψ_{**} are determined by analogy with J_{*1} and J_{**}.

Considering the nonlinearities, the expression (5.2) is modified as follows:

$$\begin{aligned} D_{*1}d(t) &= g_1(\alpha(t+1)\underline{e}(t) + (1-\alpha(t+1))\overline{e}(t)) \\ &\quad + \alpha(t+1)(D^+_{*1}\underline{d} - D^-_{*1}\overline{d}) + (1-\alpha(t+1))(D^+_{*1}\overline{d} - D^-_{*1}\underline{d}) \\ &\quad + \alpha(t+1)\tilde{x}_{*2}(t) + (1-\alpha(t+1))\hat{x}_{*2}(t) - x_{*2}(t) \\ &\quad + \alpha(t+1)C_{*1}\Psi_{*1}(y_*(t), \tilde{x}_{**}(t), y(t), u(t)) \\ &\quad + (1-\alpha(t+1))C_{*1}\Psi_{*1}(y_*(t), \hat{x}_{**}(t), y(t), u(t)) \\ &\quad - C_{*1}\Psi_{*1}(y_*(t), x_{**}(t), y(t), u(t)). \end{aligned}$$

Since the gain matrix $(g_1 \; g^T_{**})^T$ is appropriately assigned to ensure the observer's stability, we have $\tilde{x}_{**}(t) \to x_{**}(t)$ and $\hat{x}_{**}(t) \to x_{**}(t)$ and, by analogy with Section 5, $\hat{d}(t) \to d(t)$.

7. NONFULFILLMENT OF THE MATCHING CONDITION

The matching condition may often fail; the proposed approach can be still applied in this case, albeit in a more complex way. For the sake of simplicity, we consider a particular case where $k = 2$ and the term $D_*d(t)$ is included in the second equation of the linear model:

$$\begin{aligned} x_{*1}(t+1) &= x_{*2}(t) + J_{*1}y(t) + G_{*1}u(t), \\ x_{*2}(t+1) &= J_{*2}y(t) + G_{*2}u(t) + D_*d(t), \\ y_*(t) &= x_{*1}(t). \end{aligned}$$

The interval observer takes the form

$$\begin{aligned}
\underline{x}_{*1}(t+1) &= \underline{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + g_1\underline{e}(t),\\
\underline{x}_{*2}(t+1) &= J_{*2}y(t) + G_{*2}u(t) + g_2\underline{e}(t) + D_*^+\underline{d} - D_*^-\overline{d},\\
\bar{x}_{*1}(t+1) &= \bar{x}_{*2}(t) + J_{*1}y(t) + G_{*1}u(t) + g_1\overline{e}(t),\\
\bar{x}_{*2}(t+1) &= J_{*2}y(t) + G_{*2}u(t) + g_2\overline{e}(t) + D_*^+\overline{d} - D_*^-\underline{d},\\
\underline{y}_*(t) &= \underline{x}_{*1}(t), \quad \overline{y}_*(t) = \overline{x}_{*1}(t),
\end{aligned}$$

i.e., both components of the state vector are described by intervals; the variables $\underline{e}(t)$ and $\bar{e}(t)$ are determined as described above.

We find the first components of the model and the observer at the time instant $(t+2)$ by introducing time shifts in the first equations and replacing the variables $x_{*2}(t+1)$, $\underline{x}_{*2}(t+1)$, and $\overline{x}_{*2}(t+1)$ with the right-hand sides of their equations:

$$\begin{aligned}
x_{*1}(t+2) &= J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t) + D_*d(t),\\
\underline{x}_{*1}(t+2) &= J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t)\\
&\quad + g_1\underline{e}(t+1) + g_2\underline{e}(t) + D_*^+\underline{d} - D_*^-\overline{d},\\
\bar{x}_{*1}(t+2) &= J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t)\\
&\quad + g_1\overline{e}(t+1) + g_2\overline{e}(t) + D_*^+\overline{d} - D_*^-\underline{d}.
\end{aligned} \tag{7.1}$$

By analogy with (4.5), let us define

$$\alpha(t+2) = \frac{R_*y(t+2) - \underline{x}_{*1}(t+2)}{\overline{x}_{*1}(t+2) - \underline{x}_{*1}(t+2)}, \tag{7.2}$$

form the sum

$$x_{*1}(t+2) = \alpha(t+2)\overline{x}_{*1}(t+2) + (1-\alpha(t+2))\underline{x}_{*1}(t+2),$$

and replace the variables $x_{*1}(t+2)$, $\overline{x}_{*1}(t+2)$, and $\underline{x}_{*1}(t+2)$ with the right-hand sides of equations (7.1):

$$\begin{aligned}
&J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t) + D_*d(t)\\
&= \alpha(t+2)(J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t)\\
&\quad + g_1\overline{e}(t+1) + g_2\overline{e}(t) + D_*^+\overline{d} - D_*^-\underline{d})\\
&+ (1-\alpha(t+2))(J_{*1}y(t+1) + G_{*1}u(t+1) + J_{*2}y(t) + G_{*2}u(t)\\
&\quad + g_1\underline{e}(t+1) + g_2\underline{e}(t) + D_*^+\underline{d} - D_*^-\overline{d}).
\end{aligned}$$

Standard transformations yield

$$\begin{aligned}
D_*d(t) &= g_1(\alpha(t+2)\overline{e}(t+1) + (1-\alpha(t+2))\underline{e}(t+1))\\
&\quad + g_2(\alpha(t+2)\overline{e}(t) + (1-\alpha(t+2))\underline{e}(t))\\
&+ \alpha(t+2)(D_*^+\overline{d} - D_*^-\underline{d}) + (1-\alpha(t+2))(D_*^+\underline{d} - D_*^-\overline{d}),
\end{aligned} \tag{7.3}$$

and we finally arrive at the desired formula for $\hat{d}(t)$.

Remark 5. The expression (7.3) is obviously generalized to the case $k > 2$ where the term $D_*d(t)$ appears only in the last equation of the model and, considering Sections 5 and 6, to the case of its presence only in one arbitrary equation and the nonlinear case. The proposed approach is difficult to implement and simulate: it requires additional time delays for the variables $\underline{e}(t)$ and $\bar{e}(t)$ in order to obtain the expressions (7.3).

8. THE SOLUTION BASED ON THE JORDAN CANONICAL FORM

If for all solutions of equation (3.2) the variable $d(t)$ figures in two or more model components, the identification canonical form should be replaced by the Jordan diagonal form, where

$$F_* = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda_k \end{pmatrix},$$

$\lambda_1, \dots, \lambda_k$ denote the eigenvalues ($|\lambda_i| < 1$), which are supposed different. This form can always be obtained from the canonical form (3.1) [16]. As a result, the first equation in (2.3) turns into k independent equations:

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} H, \quad i = 1, \dots, k, \tag{8.1}$$

and it becomes possible to build a one-dimensional model with the variable $d(t)$. The additional requirement $\Phi L = 0$ (insensitivity to the disturbances) is considered as follows. In view of the matrix L_0 introduced above, equation (8.1) can be written as

$$(N_i \quad -J_{*i}) \begin{pmatrix} L_0(F - \lambda_i I_n) \\ H \end{pmatrix} = 0, \quad i = 1, \dots, k, \tag{8.2}$$

where I_n means an identity matrix of dimensions $(n \times n)$.

To build the model, we select an eigenvalue $\lambda = \lambda_i$ so that the matrix $\Phi = N L_0$ given by (8.2) satisfies the requirements $D_* = \Phi D \neq 0$ and $\Phi = R_* H$ for some matrix R_*.

Remark 6. The requirements $\Phi D \neq 0$ and $\Phi = R_* H$ do not necessarily hold since the exogenous disturbances represented by the matrix L_0 may restrict the set of solutions of equation (8.2) even under the matching condition. This restriction can be removed, but the resulting solution will be inexact.

The model takes the form

$$\begin{aligned} x_*(t+1) &= \lambda x_*(t) + J_* y(t) + G_* u(t) + D_* d(t), \\ y_*(t) &= x_*(t). \end{aligned} \tag{8.3}$$

Despite the one-dimensionality of the model, the technique shown in Remark 2 is inapplicable here: the right-hand side of equation (8.3) includes the variable $x_*(t)$. Therefore, it is necessary to design an interval observer described by the equations

$$\begin{aligned} \underline{x}_*(t+1) &= \lambda \underline{x}_*(t) + G_* u(t) + J_* y(t) + D_*^+ \underline{d} - D_*^- \overline{d} + g \underline{e}(t), \\ \overline{x}_*(t+1) &= \lambda \bar{x}_*(t) + G_* u(t) + J_* y(t) + D_*^+ \overline{d} - D_*^- \underline{d} + g \overline{e}(t), \\ \underline{y}_*(t) &= \underline{x}_*(t), \quad \overline{y}_*(t) = \overline{x}_*(t). \end{aligned} \tag{8.4}$$

Note that, regardless of the value of λ, the coefficient g can always be appropriately assigned to make the observer stable.

Theorem 2. *The function $d(t)$ is estimated by*

$$\begin{aligned} \hat{d}(t) = D_*^{-1} g(\alpha(t+1)\underline{e}(t) + (1 - \alpha(t+1))\overline{e}(t)) \\ + \alpha(t+1)(D_*^+ \underline{d} - D_*^- \overline{d}) + (1 - \alpha(t+1))(D_*^+ \overline{d} - D_*^- \underline{d}) \\ + \lambda(\overline{x}_*(t) - \underline{x}_*(t))(\alpha(t+1) - \alpha(t)). \end{aligned} \tag{8.5}$$

The variable $\alpha(t)$ is calculated by analogy with $\alpha(t+1)$, and the variables $\overline{x}_*(t)$ and $\underline{x}_*(t)$ are available for measurement.

Proof. The coefficient $\alpha(t+1)$ is given by (4.5), and the relation (4.7) becomes

$$\begin{gathered}\lambda x_*(t) + J_*y(t) + G_*u(t) + D_*d(t) \\ = \alpha(t+1)(\lambda\bar{x}_*(t) + G_*u(t) + J_*y(t) + D_*^+\underline{d} - D_*^-\overline{d} + g\underline{e}(t)) \\ + (1-\alpha(t+1))(\lambda\underline{x}_*(t) + G_*u(t) + J_*y(t) + D_*^+\overline{d} - D_*^-\underline{d} + g\overline{e}(t)).\end{gathered}$$

The right-hand side of this equality is transformed as follows:

$$\begin{gathered}\lambda x_*(t) + J_*y(t) + G_*u(t) + D_*d(t) = g(\alpha(t+1)\underline{e}(t) + (1-\alpha(t+1))\overline{e}(t)) \\ + \alpha(t+1)(D_*^+\underline{d} - D_*^-\overline{d}) + (1-\alpha(t+1))(D_*^+\overline{d} - D_*^-\underline{d}) \\ + J_*y(t) + G_*u(t) + \alpha(t+1)\lambda\bar{x}_*(t) + (1-\alpha(t+1))\lambda\underline{x}_*(t).\end{gathered} \tag{8.6}$$

The variable $x_*(t)$ on the left-hand side of (8.6) can be represented as

$$x_*(t) = \alpha(t)\overline{x}_*(t) + (1-\alpha(t))\underline{x}_*(t).$$

By combining it with the two last terms in (8.6), we obtain

$$\lambda(\overline{x}_*(t) - \underline{x}_*(t))(\alpha(t+1) - \alpha(t)),$$

which finally yields (8.5).

Remark 7. The described approach can also be applied to the unknown vector input $d(t)$ if the matching condition holds. In this case, the solution of equation (8.2) is found separately for each component of this vector, and the observer (8.4) is built for each component of the vector as well. Perhaps, it will be impossible to build the disturbance-insensitive observer for some components.

9. A PRACTICAL EXAMPLE

Consider a discretized model of an electric drive of the form

$$\begin{aligned} x_1(t+1) &= k_1x_2(t) + x_1(t), \\ x_2(t+1) &= k_2x_2(t) + k_3x_3(t) + \rho(t), \\ x_3(t+1) &= k_4x_2(t) + k_5x_3(t) + k_6u(t) + d(t), \\ y_1(t) &= x_1(t), \quad y_2(t) = x_3(t), \end{aligned} \tag{9.1}$$

where x_1 is the rotation angle of the gearbox output shaft; x_2 is the angular velocity of the electric motor shaft, and x_3 is the electric motor current. The coefficients $k_1, \ldots, k_6$ depend on the drive parameters and the sampling interval; the disturbance $\rho(t)$ is caused by an external load moment reduced to the motor shaft; the fault represented by the function $d(t)$ is caused by a change in the active resistance of the armature circuit, which is expressed through a corresponding change in the value of the coefficient k_5 : if $\Delta k_5(t)$ is the variation of the coefficient k_5, then $d(t) = x_3(t)\Delta k_5(t)$. This model is described by equations (2.1) with the matrices

$$F = \begin{pmatrix} 1 & k_1 & 0 \\ 0 & k_2 & k_3 \\ 0 & k_4 & k_5 \end{pmatrix}, \ G = \begin{pmatrix} 0 \\ 0 \\ k_6 \end{pmatrix}, \ H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ L = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

As is easily verified, the matching condition holds.

The disturbance-insensitive model is obtained by solving equation (3.2) with $k = 1$ and the matrices

$$R_* = \begin{pmatrix} -k_4 & k_1 \end{pmatrix}, \ J_* = \begin{pmatrix} -k_4 & k_1k_5 \end{pmatrix}, \ \Phi = \begin{pmatrix} -k_4 & 0 & k_1 \end{pmatrix}, \ G_* = k_1k_6, \ D_* = k_1.$$

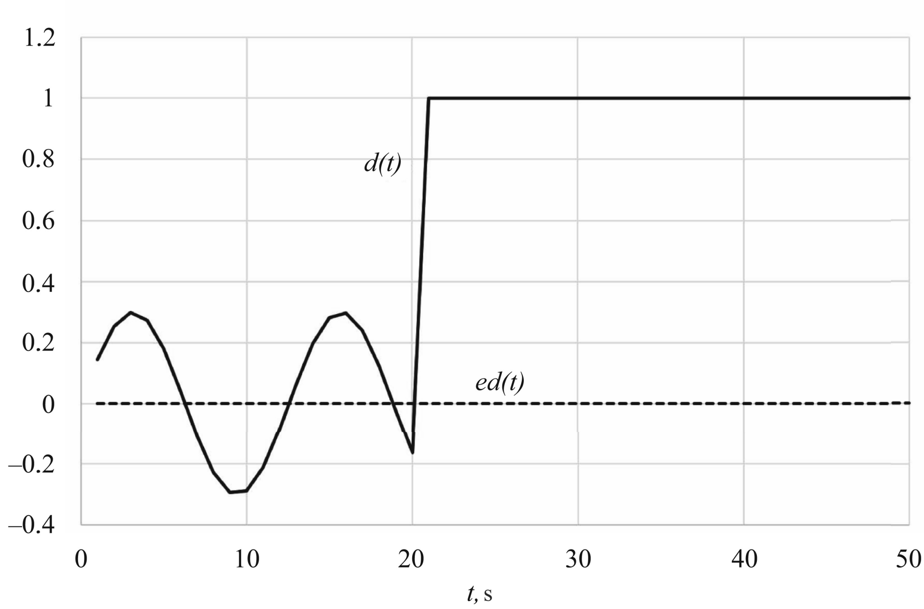

Fig. 1. The graphs of the functions $d(t)$ and $ed(t) = \hat{d}(t) - d(t)$.

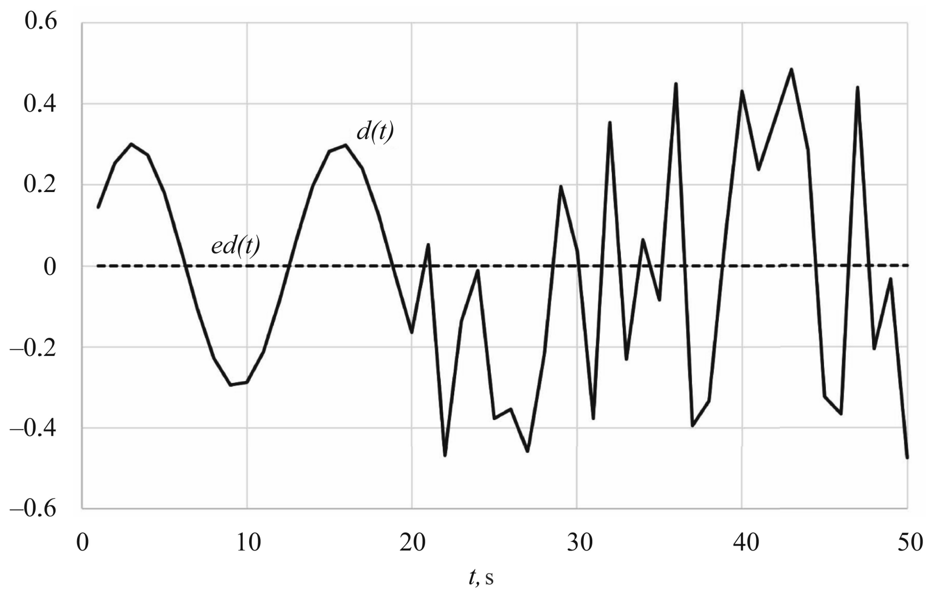

Fig. 2. The graphs of the functions $d(t)$ and $ed(t) = \hat{d}(t) - d(t)$.

The required model has the form

$$x_*(t+1) = -k_4 y_1(t) + k_1 k_5 y_2(t) + k_1 k_6 u(t) + k_1 d(t),$$
$$y_*(t) = x_*(t) = -k_4 y_1(t) + k_1 y_2(t).$$

The interval observer is described by

$$\begin{aligned} \underline{x}_*(t+1) &= -k_1 y_1(t) + k_1 k_5 y_2(t) + k_1 k_6 u(t) + k_1 \underline{d} + g\underline{e}(t), \\ \overline{x}_*(t+1) &= -k_1 y_1(t) + k_1 k_5 y_2(t) + k_1 k_6 u(t) + k_1 \overline{d} + g\overline{e}(t), \\ \underline{y}_*(t) &= \underline{x}_*(t), \quad \overline{y}_*(t) = \overline{x}_*(t), \end{aligned} \tag{9.2}$$

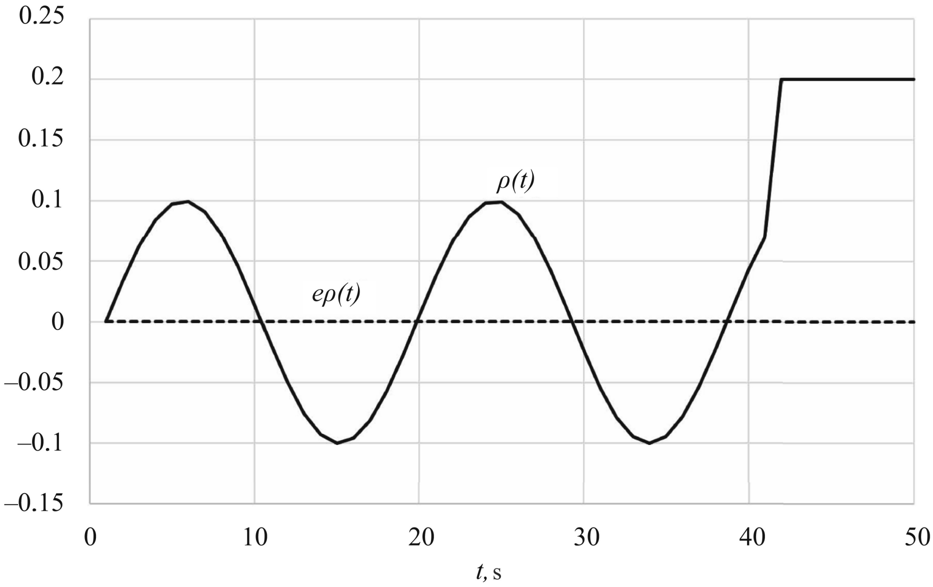

Fig. 3. The graphs of the functions $\rho(t)$ and $e\rho(t) = \hat{\rho}(t) - \rho(t)$.

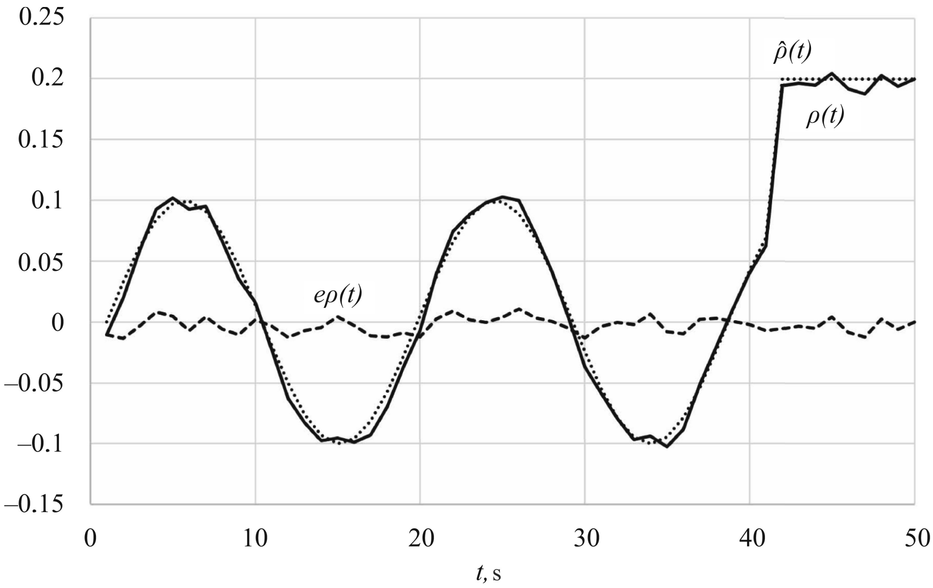

Fig. 4. The graphs of the functions $\rho(t)$, $\hat{\rho}(t)$, and $e\rho(t) = \hat{\rho}(t) - \rho(t)$.

where $\underline{e}(t) = y_*(t) - \underline{y}_*(t)$ and $\overline{e}(t) = y_*(t) - \overline{y}_*(t)$. The estimate of the variable $d(t)$ is given by (4.6).

For Matlab simulations, we choose system (9.1) and the observer (9.2) with the control input $u(t) = 0.2\sin(t/10)$, the random disturbance $\rho(t)$ with the uniform distribution on the interval $[-0.05; 0.05]$, $\underline{d} = -0.4$, $\bar{d} = 0.4$, and $g = -0.7$. The simulation results with the initial conditions $x(0) = (0\ 0\ 0)^T$, $\underline{x}_*(0) = -1$, and $\overline{x}_*(0) = 1$ are presented in Figs. 1 and 2. In particular, Fig. 1 shows the behavior of the function $d(t)$ and the estimation error $\hat{d}(t) - d(t)$ in the case $d(t) = 0.3\sin(t/2)$, $t \leqslant 20$, and $d(t) = 1$, $t > 20$; Fig. 2, the same graphs in the case $d(t) = 0.3\sin(t/2)$, $t \leqslant 20$, and the random function $d(t)$, $t > 20$, with the uniform distribution on the interval $[-0.5, 0.5]$. The estimation error is almost zero.

In addition, we estimate the unknown input $\rho(t)$ of system (9.1) under $d(t) = 0$; by assumption, $|\rho(t)| \leqslant \rho_*$, i.e., $\underline{\rho} = -\rho_*$ and $\overline{\rho} = \rho_*$. As is easily checked, the matching condition fails here. The model solving the estimation problem takes dimension 2:

$$\begin{aligned}x_{*1}(t+1) &= x_{*2}(t) + Ky_2(t),\\ x_{*2}(t+1) &= y_1(t) + (k_1k_3 - Kk_5)y_2(t) + k_1\rho(t),\\ y_*(t) &= x_{*1}(t) = y_1(t),\end{aligned}$$

where $K = k_1(1 + k_2)/k_4$. The interval observer has the form

$$\begin{aligned}\underline{x}_{*1}(t+1) &= \underline{x}_{*2}(t) + Ky_2(t) + g_1\underline{e}(t),\\ \underline{x}_{*2}(t+1) &= y_1(t) + (k_1k_3 - Kk_5)y_2(t) - k_1\rho_* + g_2\underline{e}(t),\\ \bar{x}_{*1}(t+1) &= \bar{x}_{*2}(t) + Ky_2(t) + g_1\overline{e}(t),\\ \bar{x}_{*2}(t+1) &= y_1(t) + (k_1k_3 - Kk_5)y_2(t) + k_1\rho_* + g_2\bar{e}(t),\end{aligned}$$

where $\underline{e}(t) = y_*(t) - \underline{x}_{*1}(t)$, $\overline{e}(t) = y_*(t) - \overline{x}_{*1}(t)$. The coefficient $\alpha(t + 2)$ is given by (7.2) with $R_* = (1 \;\; 0)$. The unknown input $\rho(t)$ is estimated using the expression (7.3):

$$\begin{aligned}\hat{\rho}(t) = k_1^{-1}&(g_1(\alpha(t+2)\overline{e}(t+1) + (1 - \alpha(t+2))\underline{e}(t+1))\\ &+ g_2\overline{e}(t)(\alpha(t+2)\overline{e}(t) + (1 - \alpha(t+2))\underline{e}(t))\\ &\quad + \alpha(t+2)k_1\rho_* - (1 - \alpha(t+2))k_1\rho_*).\end{aligned}$$

For Matlab simulations, we choose $\rho_* = 0.2$, $g_1 = -0.4$, and $g_2 = -0.1$. The simulation results are presented in Figs. 3 and 4, which show the behavior of the function $\rho(t)$ and the estimation error $\hat{\rho}(t) - \rho(t)$. In Fig. 3, the function $\rho(t)$ is defined by $\rho(t) = 0.3\sin(t/2)$, $t \leqslant 40$, and $\rho(t) = 1$, $t > 40$; as above, the estimation error is almost zero. In Fig. 4, additional measurement noises are introduced in the form of two independent random variables with the uniform distribution on the interval $[-0.01,\, 0.01]$. Clearly, in this case, the estimation quality gets worse.

10. CONCLUSIONS

This paper has considered discrete time-invariant systems described by linear and nonlinear dynamic models with exogenous disturbances under the so-called matching condition. Unknown inputs in such systems have been estimated based on a minimal-dimension disturbance-insensitive model of the original system. For this model, an interval observer has been designed; it produces an exact estimate of the unknown inputs in several cases. The initial solution obtained initially for the linear one-dimensional model has been extended to the multidimensional and nonlinear cases and to the systems not satisfying the matching condition. The theoretical results have been illustrated with a practical example.

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